International Journal of Theoretical Physics, Vol. 37, No. 7, 1998

General Techniques for Evaluating Twistor Diagrams

J. G. Cardoso[1]

Received April 15, 1998

Explicit differential and integral relations that afford a natural basis for systematically evaluating twistor diagrams are presented. The evaluation techniques include making use of a conformally invariant parametrization prescription along with a set of graphical computational devices which clearly bring out the striking features of the integrals associated with such diagrams. It appears that the main results of the relevant calculations carried out earlier can all be extracted from structures which arise out of effectively implementing the techniques. Both the topological character and the location of the contours presumably borne by any diagrams turn out to be specified in a unique way. Another explanation of the breaking of the conformal symmetry that arises in the context of twistor theory is automatically provided.

1. INTRODUCTION

Twistor diagrams were introduced about 26 years ago as an attempt to provide an alternative approach to pertubative quantum field theory which might produce finite amplitudes for high-energy scattering processes (Penrose and MacCallum, 1972). Presumably, this conformally invariant framework could initially afford a set of scaling-invariant multidimensional contour integrals which would take over the role of the conventional Feynman graphs for massless quantum electrodynamics in the x-representation. Explicit diagrams for the Compton and Möller processes along with other physically meaningful configurations were thus proposed. Nevertheless, the pertinent constructions had no basis that might reflect the utilization of methodical procedures except for the incorporation of the helicity-linear-momentum conservation laws and the implementation of some provisional rules associated

[1] Department of Mathematics, Centre for Technological Sciences-UDESC, Joinville 89223-100 SC, Brazil.

0020-7748/98/0700-1857$15.00/0

with a contour prescription. The actual evaluation of twistor diagrams was carried out for the first time by Qadir (1971), who showed in a notable way that the standard result for the cross section of the ϕ^4-scattering could be effectively recovered. This computational work was based upon a noninvariant parametrization prescription which did not lead to a specification of the location of the contours, with the imposition of certain constraints on the variables which partake of the integrations, and a particular choice of $SU(2, 2)$ bases having been put into effect at the intermediate stages of the calculations. Penrose likewise pointed out that processes involving the creation and annihilation of electron–positron pairs could be calculated in a contour-free fashion by defining simple (Poincaré-invariant) twistor differential operators and expressing the relevant states in the p-representation. Additionally, a transcription method was given (Penrose, 1975) whereby covariant kinematical quantities carrying massless fields may be expressed in terms of diagrams, but no specific rules for translating the underlying spacetime configurations were deduced. The situation concerning the location and topological definition of the contours was investigated on the basis of the application of an apparently rigorous mathematical treatment of the diagrams (Sparling, 1975). In this connection, it was stated later that the original diagram for the Möller process bears an infrared divergence which can be regularized by ascribing compact bounded contours to the corresponding integral and utilizing a nonprojective shifting technique (Hodges and Huggett, 1980; Hodges, 1983a–c). An interesting result obtained more recently has suggested that cohomological contours for the so-called elementary-state branches must necessarily carry S^1-pieces (Huggett, 1993).

In the present work, we provide a basis for systematically evaluating twistor diagrams in the x-representation. Our methods involve making use of a system of differential and integral relations which particularly give rise to a useful pole-order lowering technique. It will be shown that such relations enable one to perform several integrations at a time, thereby reducing any conformally invariant diagrams to integrals of wedge products between Poincaré-invariant one-forms. It will become clear that the breaking of the conformal symmetry is due essentially to an intrinsic differential property of the integrands of the integrals which represent the diagrams. In each case, the completion of the evaluation is accomplished by introducing a new parametrization prescription which immediately yields elementary expressions possessing a spacetime meaning. It appears that the conformal invariance can be recovered by following up suitable integration procedures and allowing for appropriate identifications involving auxiliary twistors. Accordingly, we will see that the contours must be thought of as the topological product of S^1's which are always unambiguously located in products of complex planes. We also construct a set of graphical devices that help keep track of the index blocks which occur in the calculation of any of the diagrams considered hitherto. In order to illustrate the principal characteristics of the methods, it will be enough to evaluate explicitly only some of the (projective) configurations taken up by Qadir (1978). It will turn out, in effect, that all the final results of the calculations carried out earlier can apparently be extracted from structures which arise out of implementing the techniques.

The organization of the paper is as follows. In Section 2, we exhibit our system of relations, the description of the key integration processes being given thereafter. We build up the graphs in Section 3. For completeness, the explanation of the basic rules underlying the construction will be included. Section 4 deals with the illustration of the methods. There, we will make further observations on the integrations and a few particular points regarding the appropriateness of the identifications without touching upon the physical aspects of the diagrams. Some remarks on our work will be made in Section 5. No attempt to provide any cohomological interpretation of our statements will be made herein. The usual twistor-diagram conventions and rules will from now on be taken for granted. All the elementary twistors entering the diagrams are at the outset assumed to be null and defined in real Minkowski space **RM**. The π-parts that occur in each infinity-twistor inner product will *not* be subject to any proportionality relations. The set of complex numbers will be denoted by **C**. For the ordinary differential forms on the projective twistor spaces, we have the notation

$$\delta A = I_{\mu\nu} A^{\mu} dA^{\nu}$$

$$\delta X = I^{\mu\nu} X_{\mu} dX_{\nu}$$

and

$$\Delta A = \frac{1}{3!}\,\varepsilon_{\mu\nu\lambda\sigma} A^{\mu} dA^{\nu} \wedge dA^{\lambda} \wedge dA^{\sigma}$$

$$\Delta X = \frac{1}{3!}\,\varepsilon^{\mu\nu\lambda\sigma} X_{\mu} dX_{\nu} \wedge dX_{\lambda} \wedge dX_{\sigma}$$

The nonprojective forms are written as

$$d^2 A = \frac{1}{2}\, I_{\mu\nu} dA^{\mu} \wedge dA^{\nu}$$

$$d^2 X = \frac{1}{2}\, I^{\mu\nu} dX_{\mu} \wedge dX_{\nu}$$

and

$$d^4A = \frac{1}{4!}\,\varepsilon_{\mu\nu\lambda\sigma}dA^{\mu} \wedge dA^{\nu} \wedge dA^{\lambda} \wedge dA^{\sigma}$$

$$d^4X = \frac{1}{4!}\,\varepsilon^{\mu\nu\lambda\sigma}dX_{\mu} \wedge dX_{\nu} \wedge dX_{\lambda} \wedge dX_{\sigma}$$

Wedge products between projective forms of the same degree will be denoted as

$$\delta AX \cdots = \delta A \wedge \delta X \wedge \cdots$$

and

$$\Delta AX \cdots = \Delta A \wedge \Delta X \wedge \cdots$$

In general, pieces of statements carrying twistors of one type have counterparts carrying twistors of the other type. Either element of a pair of such dual structures can at once be derived from the other by replacing kernel letters and invoking the canonical twistor pseudo-Hermitian conjugation rule (Penrose, 1967; Penrose and Rindler, 1986)

Upper index		Lower index
0	↔	2
1	↔	3
2	↔	0
3	↔	1

For this reason, we will not spell out the entire system of relations when developing Sections 2 and 3.

2. DIFFERENTIAL AND INTEGRAL RELATIONS

To start with, let us write down the simple relations

$$d^2X = \frac{d\zeta}{\zeta} \wedge \delta X = \frac{1}{2}\,d\delta X \tag{2.1}$$

and

$$d^4X = \frac{d\eta}{\eta} \wedge \Delta X = \frac{1}{4}\,d\Delta X \tag{2.2}$$

with ζ standing for either of the components (X_0, X_1) and η being any of the components $\{X_\mu\}$. Loosely speaking, the structures (2.1) and (2.2) enable us to pass from nonprojective integrals to projective ones by performing trivial integrations. We also have the device

$$(A^{\mu} dX_{\mu}) \wedge \Delta X = (A^{\mu} X_{\mu}) d^4 X \tag{2.3}$$

A short calculation yields the Poincaré-invariant relation

$$(A^{\lambda} dX_{\lambda}) \wedge (B^{\sigma} dX_{\sigma}) \wedge d^2 X = (I_{\lambda\sigma} A^{\lambda} B^{\sigma}) d^4 X \tag{2.4}$$

the corresponding projective statement being written out explicitly as

$$\begin{aligned}(A^{\lambda} dX_{\lambda}) &\wedge (B^{\sigma} dX_{\sigma}) \wedge \delta X \\ &= (I_{\lambda\sigma} A^{\lambda} B^{\sigma}) \Delta X + [(B^{\sigma} X_{\sigma})(A^{\lambda} dX_{\lambda}) - (A^{\lambda} X_{\lambda})(B^{\sigma} dX_{\sigma})] \wedge \frac{d\zeta}{\zeta} \wedge \delta X\end{aligned} \tag{2.5}$$

We will see in a moment that the second term on the right-hand side of (2.5) vanishes identically.

In fact, there is a two-form analogue of (2.3) which can be derived in a transparent manner by parametrizing

$$X_{\mu} = \alpha E_{\mu} + \beta F_{\mu} \tag{2.6}$$

where α and β both belong to $\mathbf{C} - \{0\}$, and (E_{μ}, F_{μ}) is a pair of fixed (null) twistors. We thus have

$$d^2 X = (I^{\rho\sigma} E_{\rho} F_{\sigma}) d\alpha \wedge d\beta \tag{2.7a}$$

along with

$$(C^{\mu} dX_{\mu}) \wedge \delta X = (C^{\mu} X_{\mu}) d^2 X \tag{2.7b}$$

such that

$$\delta X = (I^{\rho\sigma} E_{\rho} F_{\sigma}) \alpha^2 d\xi \tag{2.7c}$$

with $\xi = \beta/\alpha$. It should be emphatically observed that (2.7) can be incorporated into the left-hand side of (2.5) only through either $(A^{\lambda} dX_{\lambda})$ or $(B^{\sigma} dX_{\sigma})$ since the implementation of the structures for both one-forms would give rise to a useless relation involving a vanishing three-form. All the integral expressions to be derived in this section allow the combination of this prescription with (2.2) and (2.3). The consistency of (2.2) and (2.4) with (2.5) thus stems from

$$\begin{aligned}&[(B^{\sigma} X_{\sigma})(A^{\lambda} dX_{\lambda}) - (A^{\lambda} X_{\lambda})(B^{\sigma} dX_{\sigma})] \wedge \frac{d\zeta}{\zeta} \wedge \delta X \\ &\quad = [(A^{\lambda} X_{\lambda})(B^{\sigma} X_{\sigma}) - (B^{\sigma} X_{\sigma})(A^{\lambda} X_{\lambda})] \frac{d\zeta}{\zeta} \wedge \frac{d\zeta'}{\zeta'} \wedge \delta X \equiv 0\end{aligned} \tag{2.8}$$

where ζ' is defined in the same way as ζ. By virtue of the nullity of X_μ, the twistors E_μ and F_μ can be considered as two generators of the null cone of a fixed point of **RM**, whence, if $f(X_\tau)$ is a homogeneous meromorphic function of degree -2, then upon calling upon (2.1), we are led to the formal statement

$$\int_{\gamma_X} f(X_\tau) \frac{(C^\mu dX_\mu)}{(C^\mu X_\mu)} \wedge \delta X = 2\pi i \int_{\gamma_\xi} (I^{\rho\sigma} E_\rho F_\sigma) f(E_\tau + \xi F_\tau) d\xi \quad (2.9)$$

where $\gamma_X = \gamma_\zeta \times \gamma_\xi$, with γ_ζ and γ_ξ being oriented S^1-contours which suitably surround the singularities lying in the $\zeta\xi$-planes. For example, up to choices of orientation, we have

$$\frac{1}{(2\pi i)^2} \int_{\gamma_X} \frac{(C^\mu dX_\mu) \wedge \delta X}{(I^{\rho\sigma} M_\rho X_\sigma)(C^\mu X_\mu)^2} = \frac{(I^{\rho\sigma} E_\rho F_\sigma)}{2(I^{\alpha\beta} M_\alpha E_{[\beta})(C^\delta F_{\delta]})} \quad (2.10)$$

and (Hughston, 1979)

$$\frac{1}{2\pi i} \int_{\gamma_X} \frac{\delta X}{(I^{\rho\sigma} U_\rho X_\sigma)(I^{\mu\nu} V_\mu X_\nu)} = \frac{1}{(I^{\alpha\beta} U_\alpha V_\beta)} \quad (2.11)$$

where M_μ, U_μ, and V_μ are auxiliary twistors, with the relation

$$\begin{aligned}(I^{\rho\sigma} E_\rho U_\sigma)(I^{\mu\nu} F_\mu V_\nu) - (I^{\rho\sigma} E_\rho V_\sigma)(I^{\mu\nu} F_\mu U_\nu) \\ = (I^{\rho\sigma} E_\rho F_\sigma)(I^{\mu\nu} U_\mu V_\nu) \quad (2.12)\end{aligned}$$

having also been taken into account. In (2.11), γ_X is particularly an S^1 that surrounds in the ξ plane either of the singularities

$$a = (-1) \frac{(I^{\rho\sigma} E_\rho U_\sigma)}{(I^{\mu\nu} F_\mu U_\nu)}, \qquad b = (-1) \frac{(I^{\rho\sigma} E_\rho V_\sigma)}{(I^{\mu\nu} F_\mu V_\nu)}$$

Let now $\mathscr{F}$ denote symbolically a twistor function whose homogeneity degrees are prescribed in such a way that any integrands bearing $\mathscr{F}$ will remain invariant when all the variables involved undergo arbitrary scalings. We thus consider the simple-pole branch

$$\begin{aligned}\int_{\Gamma_{XW\ldots Z}} \mathscr{F} \frac{d(G^\alpha X_\alpha)}{(G^\alpha X_\alpha)} \wedge \frac{d(H^\beta X_\beta)}{(H^\beta X_\beta)} \wedge \delta X \wedge \Delta W \ldots Z \\ = (2\pi i)^2 \int_{\gamma_{XW\ldots Z}} \mathscr{F}\, \delta X \wedge \Delta W \ldots Z \quad (2.13)\end{aligned}$$

where $\Gamma_{XW\ldots Z} = \mathscr{C}_G \times \mathscr{C}_H \times \gamma_{XW\ldots Z}$, with $\mathscr{C}_K$ being an S^1 of either orientation that surrounds the origin of the λ_K plane and λ_K denoting either $G^\alpha X_\alpha$ or $H^\beta X_\beta$. Obviously, $\gamma_{XW\ldots Z} = \gamma_X \times \Gamma_{W\ldots Z}$, with $\Gamma_{W\ldots Z}$ playing a symbolic role at this stage. In case both G^μ and H^μ are constant twistors, the expression (2.5) yields at once

$$\int_{\Gamma_{XW\ldots Z}} \mathcal{F} \frac{(I_{\alpha\beta} G^\alpha H^\beta)}{(G^\alpha X_\alpha)(H^\beta X_\beta)} \Delta XW \ldots Z$$
$$= (2\pi i)^2 \int_{\gamma_{XW\ldots Z}} \mathcal{F} \, \delta X \wedge \Delta W \ldots Z \qquad (2.14)$$

If either ΔG or ΔH occurs in (2.13), or even if both the forms enter the integrals, some manipulations using appropriately the relations (2.1)–(2.3), (2.5), and (2.7b) together with a suitable selection of contours for the $\zeta\eta$ integrations yield the structure (2.14) once again up to an ordering sign [see (2.17) below].

A procedure similar to the one just completed produces a clear prescription for lowering the orders of multiple-pole branches. We have, in effect,

$$\int_{\Gamma_{XW\ldots Z}} \mathcal{F} \frac{d(G^\alpha X_\alpha)}{(G^\alpha X_\alpha)^m} \wedge \frac{d(H^\beta X_\beta)}{(H^\beta X_\beta)^n} \wedge \delta X \wedge \Delta W \ldots Z = 0 \qquad (2.15)$$

with $m \geq 2$ and $n \geq 1$. It is evident that the case where $dG^\mu = 0 = dH^\mu$ is useless. If only G^μ varies, say, we obtain upon expanding the left-hand side of (2.15)

$$\int_{\Gamma_{XW\ldots Z}} \mathcal{F} \frac{[(X_\alpha dG^\alpha) + (G^\alpha dX_\alpha)]}{(G^\alpha X_\alpha)^m (H^\beta X_\beta)^n} \wedge (H^\beta dX_\beta) \wedge \delta X \wedge \Delta WG \ldots Z$$
$$= \int_{\Gamma_{XW\ldots Z}} \mathcal{F} \frac{(I_{\alpha\beta} G^\alpha H^\beta)}{(G^\alpha X_\alpha)^m (H^\beta X_\beta)^n} \Delta XWG \ldots Z$$
$$- (2\pi i)^2 \int_{\gamma_{XW\ldots Z}} \mathcal{F} \frac{\delta X \wedge \Delta WG \ldots Z}{(G^\alpha X_\alpha)^{m-1} (H^\beta X_\beta)^{n-1}}$$
$$+ \int_{\Gamma_{XW\ldots Z}} \mathcal{F} \frac{(G^\mu dX_\mu) \wedge (d\zeta/\zeta) \wedge \delta X \wedge \Delta WG \ldots Z}{(G^\alpha X_\alpha)^m (H^\beta X_\beta)^{n-1}}$$
$$- \int_{\Gamma_{XW\ldots Z}} \mathcal{F} \frac{(H^\mu dX_\mu) \wedge (d\zeta/\zeta) \wedge \delta X \wedge \Delta WG \ldots Z}{(G^\alpha X_\alpha)^{m-1} (H^\beta X_\beta)^n} = 0 \qquad (2.16)$$

whence

$$\int_{\Gamma_{XW\ldots Z}} \mathscr{F} \frac{(I_{\alpha\beta} G^{\alpha} H^{\beta})}{(G^{\alpha} X_{\alpha})^{m} (H^{\beta} X_{\beta})^{n}} \, \Delta XWG \ldots Z$$

$$= (2\pi i)^2 \int_{\gamma_{XW\ldots Z}} \mathscr{F} \frac{\delta X \wedge \Delta WG \ldots Z}{(G^{\alpha} X_{\alpha})^{m-1} (H^{\beta} X_{\beta})^{n-1}} \tag{2.17}$$

Letting both G^{μ} and H^{μ} vary yields the same result as (2.17), but now the product of Δ-forms carries also ΔH.

The relations derived above tell us that we can reduce integrals of projective three-forms to integrals of projective one-forms by performing integrations along well-prescribed S^1-contours. Had we instead worked out inhomogeneous structures, a similar reduction would have been naturally brought about. Hence, the overall contours borne by any multidimensional integrals can be fibered into S^1-factors. In practice, (2.14) is most naively used when the functional dependence of $\mathscr{F}$ on X_{μ} involves only simple-pole elementary states and (G^{μ}, H^{μ}) is a pair of fixed auxiliary twistors. In such situations, the singularities carrying G^{μ} and H^{μ} are evidently represented by external lines. The application of (2.14) to cases where at least one of G^{μ} and H^{μ} is to be integrated gives rise to a disconnection of the internal branches of the diagrams being dealt with, which may induce the occurrence of divergent results if the parametrization (2.6) is not taken to carry conjugate variable twistors. Inserting this kind of twistor into (2.6) does indeed restore the connectedness, but this procedure actually causes a loss of holomorphicity. The recovery of the holomorphicity can be attained only if suitably balanced dualization relations are brought into the pertinent integrands. What happens is, in effect, that the factors which spoil the holomorphicity are canceled when we implement adequate dualizations. In contrast to simple-pole patterns, multiple poles borne by standard elementary states are associated with internal lines, but higher order poles of configurations that carry functional branches usually appear as external lines. In typical elementary-state cases when $n \geq 2$, the recovery of the conformal invariance will be ensured if we utilize (2.17) upon performing integrations leading to δ-forms. In these cases, the order-lowering devices also enable us to avoid dealing with vanishing integrals of the type $\int_{\gamma_{\zeta}} (\zeta - a)^{-N} \, d\zeta$ with $N \geq 2$, the use of (2.17) for $n = 1$ being often made when the integrals under consideration carry constant H^{β}-twistors. Frequently, in treating integrals which bear simple and multiple poles together with functional branches we may initially retain the higher order pole pieces and then use the theorem of residues along with elementary derivative techniques for completing the evaluations at issue. In each of the classical cases of branches of logarithmic functions, for instance, this procedure entails eliminating at least one of those functions at the final stages of the relevant calculations. Some further points concerning the structure of (2.14) and (2.17) will be made in Sections 4 and 5.

3. GRAPHICAL DEVICES

One of the main facts upon which the construction of the usual graphical computational devices for decomposable tensors rests is that any symmetry operation can be looked upon as either an index or a kernel-letter permutation rule. Here, the indices of any tensors are represented by vertical lines contained in the same plane. The index lines lie above or beneath the kernel letters according to whether the corresponding indices are upper or lower. Diagram symbols may be used in place of letters, in which case the index lines are taken to start at points of the symbols. The twistor conjugation amounts in any case to a reflection of the configurations in some horizontal. Index contractions are implemented by joining endpoints of index lines. Outer products are represented by juxtaposing the structures that correspond to the relevant factors. Skew-symmetrizations and symmetrizations are denoted, respectively, by horizontal straight and staggered lines which cross the index lines associated with the indices taken up by the operations, but such an operation will henceforth be regarded as a kernel-letter rule. Suitable displacements of branches leave the graphs invariant in the sense that the former and latter configurations appear to represent the same tensors.

For twistors $A^{\beta}, \ldots, D^{\beta}$ and $X_{\alpha}, \ldots, V_{\alpha}$, we thus have the representation

$$A^{\beta} = \overset{|}{A}, \ldots, \quad D^{\beta} = \overset{|}{D} \tag{3.1a}$$

and

$$X_{\alpha} = \overset{X}{|}, \ldots, V_{\alpha} = \overset{V}{|} \tag{3.1b}$$

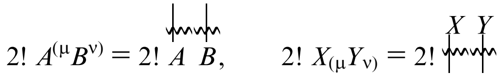
The inner product of A^{β} with X_{α}, say, is depicted as

$$A^{\mu} X_{\mu} = \overset{X}{\underset{A}{|}} \tag{3.2}$$

Additionally, we have the representations

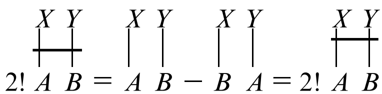
$$2!\, A^{[\mu}B^{\nu]} = 2!\, \overset{\vdash\!\!\dashv}{A\;\;B}, \qquad 2!\, X_{[\mu}Y_{\nu]} = 2!\, \overset{X\;\;Y}{\vdash\!\!\dashv} \tag{3.3}$$

and



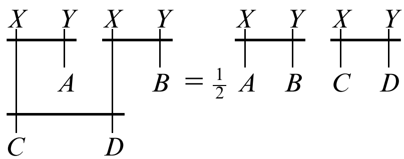
(3.4)

It is obvious that we can also depict structures similar to (3.3) and (3.4), which involve three and four twistors. As far as Section 4 is concerned, the symmetric case will occur only in the evaluation of one of the diagrams. Thus, we shall concentrate our attention upon skew-symmetric configurations, but a symmetrized-block equality which plays an auxiliary role in the evaluation referred to above will be constructed. We have the simple defining expansion



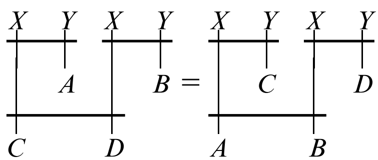
(3.5)

which provides us with the relations



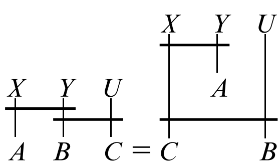
(3.6a)

and



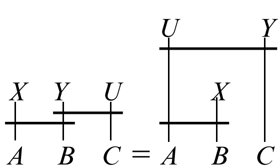
(3.6b)

along with



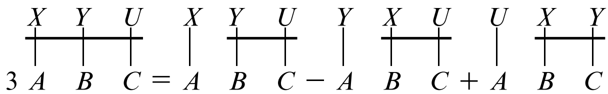
(3.6c)

and



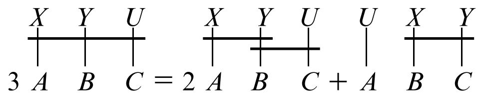
(3.6d)

One of the contracted three-twistor expansions turns out to be expressed as

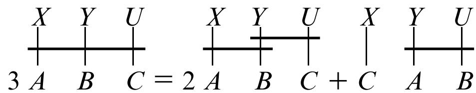


(3.7)

and thus suggests writing the useful statements

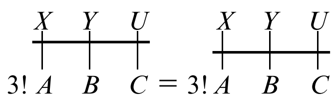


(3.8)



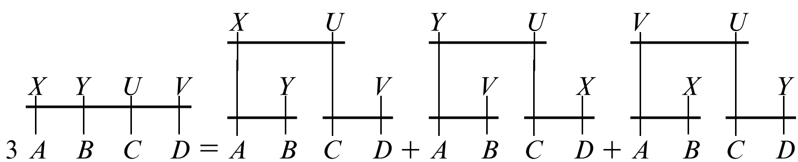
(3.9)

and



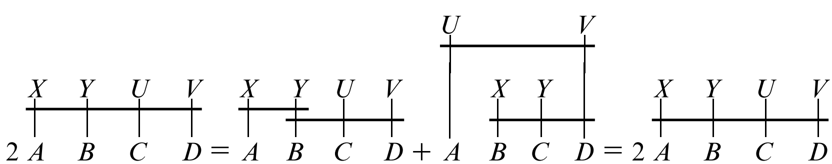
(3.10)

We also have



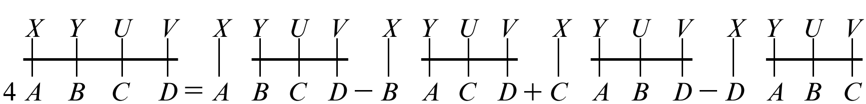
(3.11)

along with



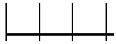
(3.12)

and

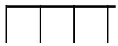


(3.13)

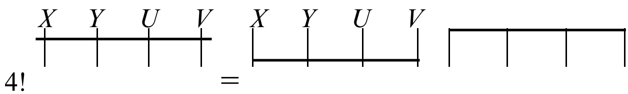
The alternating twistors are represented by

$\varepsilon^{\alpha\beta\gamma\delta} =$  (3.14)

and

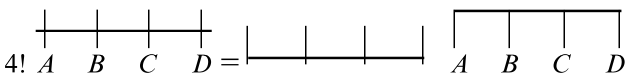
$\varepsilon_{\mu\nu\lambda\sigma} =$  (3.15)

In the canonical bases, (3.14) and (3.15) can be particularly employed to express totally skew-symmetric four-twistor blocks according to the schemes



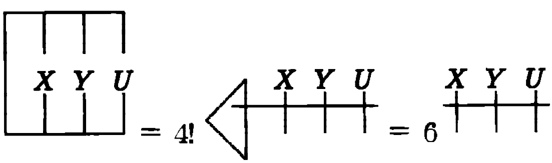
(3.16a)

and

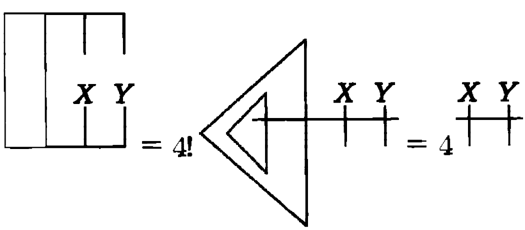


(3.16b)

We stress that any eventual contractions with (3.16) must be made in an ordered manner. For lower-index twistors, for instance, straightforward computations thus yields

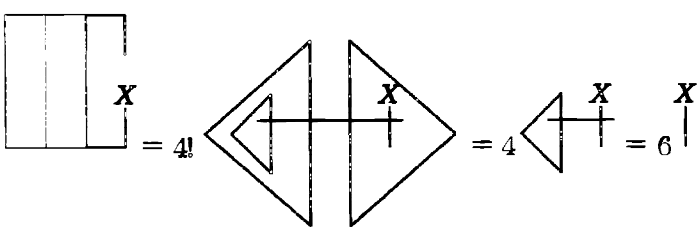


(3.17)



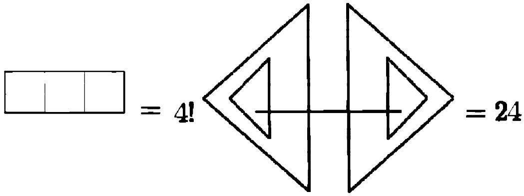
(3.18)

and



(3.19)

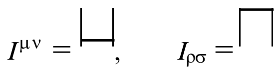
together with



(3.20)

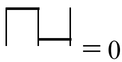
with the bracketed lines meaning index contractions.

It is convenient to introduce a representation for the infinity twistors. We have the graphs

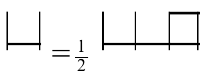


(3.21)

which are subject to

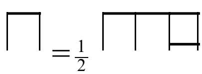


(3.22)



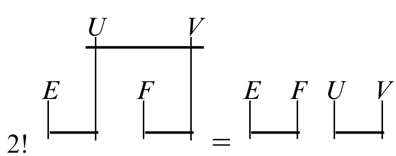
(3.23a)

and



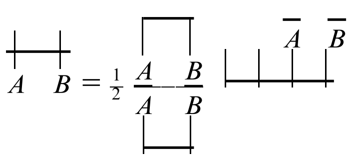
(3.23b)

For the property (2.12), we accordingly have the configuration



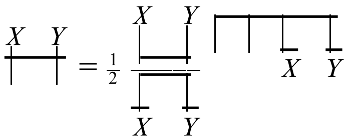
(3.23c)

The standard dualization relations are thus depicted as



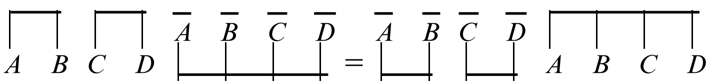
(3.24a)

and



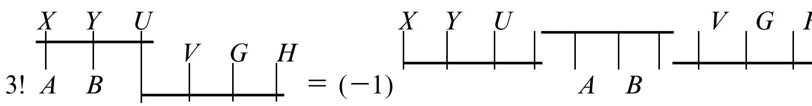
(3.24b)

Usually, the **RM**-lines associated with the elementary twistors which enter into each of the above dualizations are required to meet such that (3.5) normally carries information on the distance between the vertices of two null cones in **RM**. We are then led to the reality statement



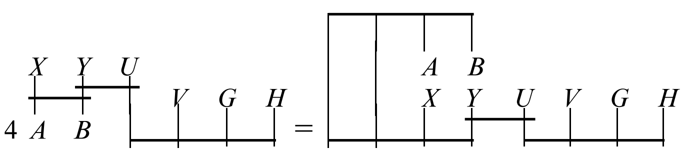
(3.24c)

A set of especially useful structures can be constructed by utilizing the relations (3.17)–(3.19). We have, for instance,



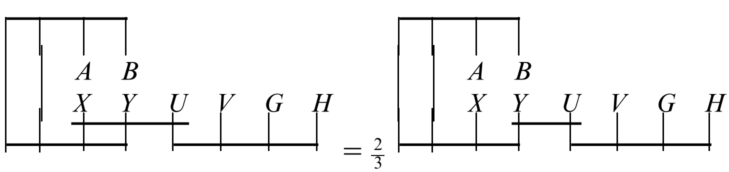
(3.25)

and



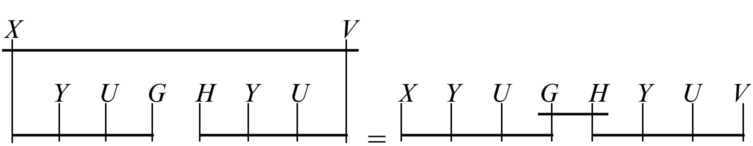
(3.26)

Supposing here for once that the piece $\overset{X}{\underset{C}{|}}$ of (3.9) vanishes, we can write



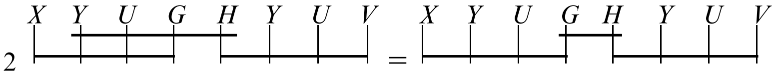
(3.27)

where (3.10) has been employed. In addition, we have the skew-symmetry property



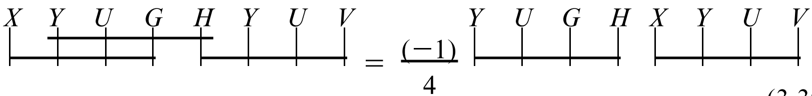
(3.28)

Carrying out an expansion of the type (3.12) yields



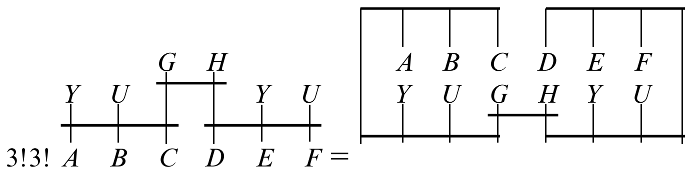
(3.29)

We can disconnect the left-hand side of (3.29) by accounting for (3.16) and (3.19). We thus have



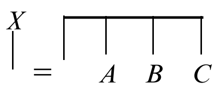
(3.30)

Making particular use of (3.17), we obtain



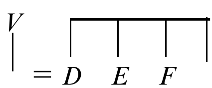
(3.31)

It should be noticed that (3.29) and (3.30) can be recovered from (3.31) by defining



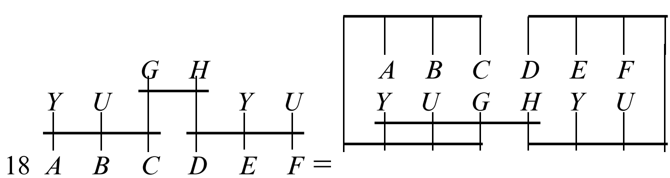
(3.32a)

and



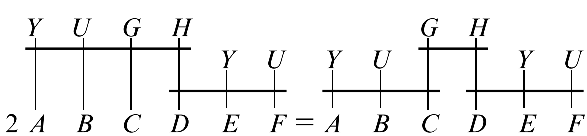
(3.32b)

For instance,



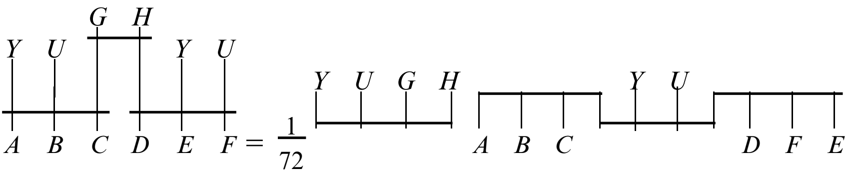
(3.33)

It follows that



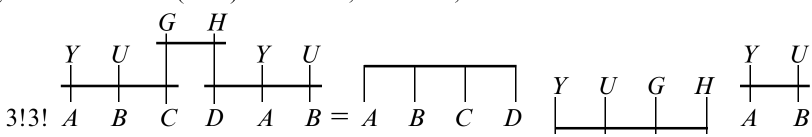
(3.34)

and, consequently, we can write



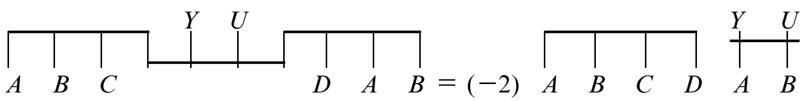
(3.35)

Identifying $A \equiv E$ and $B \equiv F$ splits out the connected ε-piece that occurs on the right-hand side of (3.35). We have, in effect,



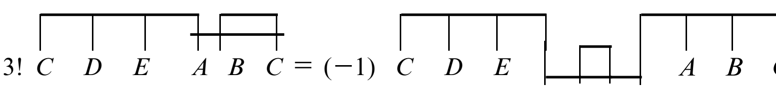
(3.36)

If we combine (3.35) and (3.36), recalling (3.16), we arrive at the simplified relation



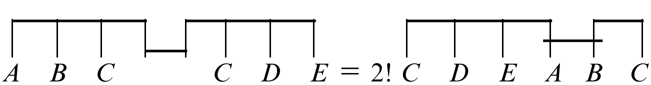
(3.37)

It is also useful to consider the Poincaré-invariant structure



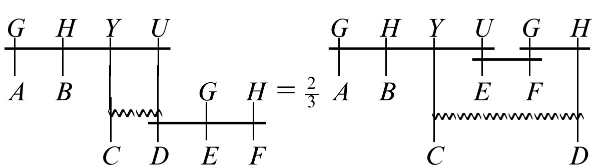
(3.38)

whence, expanding its left-hand side and using (3.23a) gives



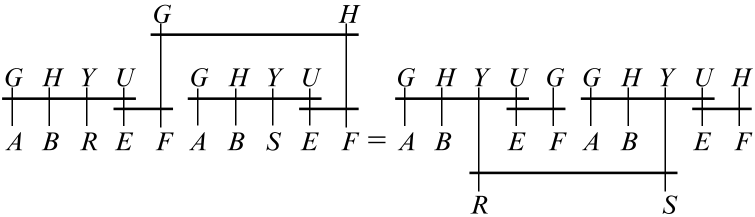
(3.39)

Let us now symmetrize the structure (3.34) over C and D. Invoking (3.9) and interchanging kernel letters, we readily obtain



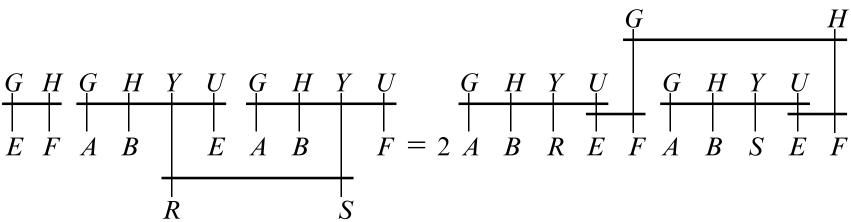
(3.40)

which also holds when $C \equiv D$. It is of interest to write down the structure which arises from the right-hand side of (3.40) when we drop the staggered line and replace D^{α} with a kernel of the same type as the connected one that involves skew-symmetrizations over *GHYU* and *EF*. We thus have the property



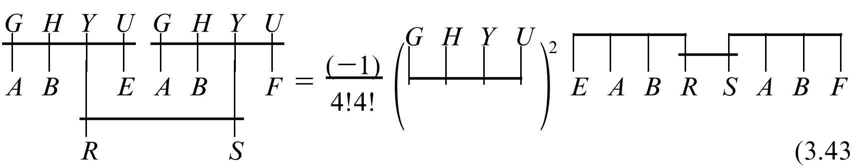
(3.41)

which can be rapidly established by displacing branches adequately. Carrying out the expansions in E and F entails the simplification



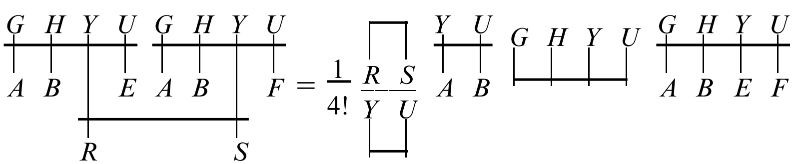
(3.42)

since the latter *RS*-connected piece is also skew in *EF*. Hence, incorporating (3.16) into the left-hand side of (3.42) implies that



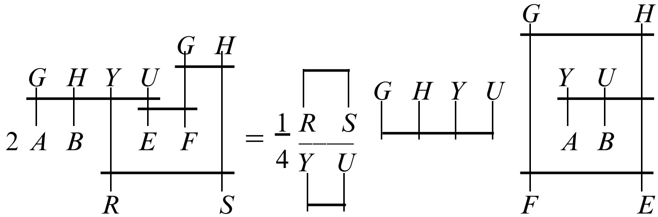
(3.43)

Moreover, taking (R^{β}, Y_{α}) and (S^{β}, U_{α}) as pairs of conjugate twistors and making use of (3.37) yields



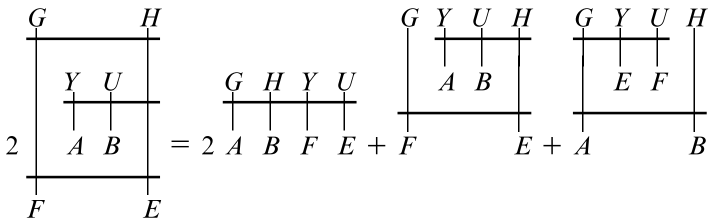
(3.44)

Under these circumstances, we also obtain



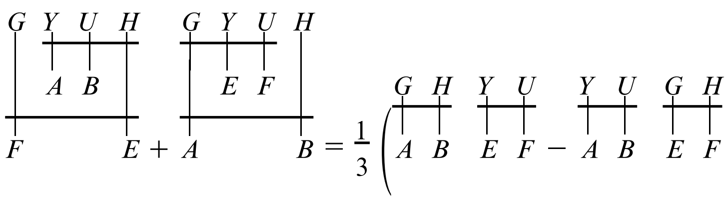
(3.45)

with



(3.46)

and



(3.47)

4. EXPLICIT EVALUATION OF DIAGRAMS

A remarkable feature of Penrose's diagrams is related to the fact that all the variable twistors involved in the representative integrals are taken to be independent of each other. Thus, the integration of one spotted vertex can be carried out as if the twistors occurring in the other vertices were fixed. When combined together with the scaling-invariance property, this feature allows us to neatly implement the techniques of the foregoing sections in connection with our immediate purposes here. Upon carrying out calculations that reduce to δ-integrals connected diagrams possessing only simple poles, we will pick up singularity pieces associated with inner products involving both constant and variable *GH*-twistors. Indeed, the cases in which the explicit denominators of (2.14) represent external and internal lines will both take place. The relation (2.17) for $n = 1$ will be applied strictly to integrals that do not carry ΔH. It will be seen that the recovery of some of the results mentioned earlier generally requires identifications which involve twistors of the same type and conjugate twistors. As regards simple-pole diagrams, dualization identifications appear to bear a "stronger" character in that standard structures can be brought back in a more effective way. Whenever the dualization prescription is applied to the auxiliary twistors of (2.6), the inner products $G^{\alpha}X_{\alpha}$ and $H^{\beta}X_{\beta}$ of (2.14) turn out to vanish identically as the integrals of (2.13) which are taken along $\mathscr{C}_K$ automatically select X_{μ} at the origins of the λ_K planes. In these situations, the location of vertices of null cones is clearly dependent upon the choice of G^{β} and H^{β}, although the blocks that afford the **RM** values of the diagrams remain essentially unaffected when interchanges of kernel letters for constant twistors of the same valence are allowed for. In the first instance, making such interchanges merely entails a change of dummy integration variables. The above properties are also borne by the ordinary configurations that carry logarithms, but dualizations seriously fail to hold for the elementary states that carry one double pole. Some higher order elementary states admit identifications which involve mixing up and interchanging kernel letters of twistors that occur in different branches.

In what follows, the integration procedures will be particularly illustrated. As the formulas (2.14), (2.17), (3.24a), and (3.24b) will be used so many times, we shall no longer refer to them explicitly. We will adopt the additional convention according to which each projective spotted vertex contributes a $(2\pi i)^{-3}$ factor to an integral. The (overall) signs corresponding to choices of contour orientations will be ignored.

Let us begin with the simple diagram exhibited in Fig. 1. The associated integral is

$$I_1 = \frac{1}{(2\pi i)^6} \int_{\Gamma_{WZ}} \frac{\Delta WZ}{\begin{matrix} W & W & W & W & D & E & F \\ | & | & | & | & | & | & | \\ A & B & C & Z & Z & Z & Z \end{matrix}} \tag{4.1}$$

Performing integrations and parametrizing

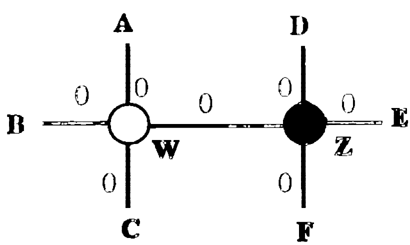


Fig. 1. A two-spotted-vertex tree diagram. The relevant singularities are all simple poles.

$$W_\alpha = L_\alpha + \mu M_\alpha \tag{4.2}$$

yields, say,

$$I_1 = \frac{1}{(2\pi i)^2} \int_{\gamma_\mu \times \gamma_Z} \frac{\overset{L\;M}{\sqcup}\, d\mu \wedge \delta Z}{\underset{A\;B}{\sqcap}\;\overset{E\;F}{\sqcup}\;\begin{smallmatrix} M \\ | \\ C \end{smallmatrix}\;\begin{smallmatrix} M \\ | \\ Z \end{smallmatrix}\;\begin{smallmatrix} D \\ | \\ Z \end{smallmatrix}\;(\mu - a)(\mu - b)} \tag{4.3}$$

where

$$a = (-1)\, \begin{smallmatrix} L \\ | \\ C \end{smallmatrix} \Big/ \begin{smallmatrix} M \\ | \\ C \end{smallmatrix} \tag{4.4a}$$

and

$$b = (-1)\, \begin{smallmatrix} L \\ | \\ Z \end{smallmatrix} \Big/ \begin{smallmatrix} M \\ | \\ Z \end{smallmatrix} \tag{4.4b}$$

Now, carrying out the μ-integral and parametrizing

$$Z^\beta = G^\beta + \nu H^\beta \tag{4.5}$$

we obtain the expression

$$I_1 = \frac{\underset{G\;H}{\sqcap}\;\overset{L\;M}{\sqcup}}{2!\;\underset{A\;B}{\sqcap}\;\overset{E\;F}{\sqcup}\;\begin{array}{cc} L & M \\ \hline C & H \end{array}\;\begin{smallmatrix} D \\ | \\ H \end{smallmatrix}} \; \frac{1}{2\pi i} \int_{\gamma_\nu} \frac{d\nu}{(\nu - A)(\nu - B)} \tag{4.6}$$

with

$$A = (-1)\, \begin{smallmatrix} D \\ | \\ G \end{smallmatrix} \Big/ \begin{smallmatrix} D \\ | \\ H \end{smallmatrix} \tag{4.7a}$$

and

$$B = (-1)\, \begin{array}{cc} L & M \\ \hline C & G \end{array} \Big/ \begin{array}{cc} L & M \\ \hline C & H \end{array} \tag{4.7b}$$

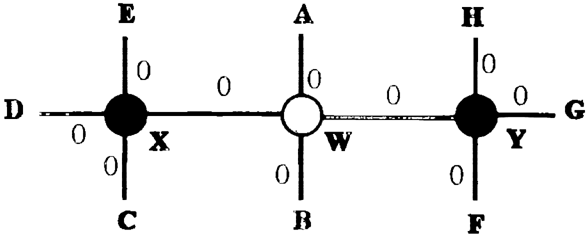


Fig. 2. A tree carrying three spotted vertices and simple poles.

It follows that

$$I_1 = \frac{\overline{\begin{matrix}G & H\end{matrix}}\ \underline{\begin{matrix}L & M\end{matrix}}}{4\ \overline{\begin{matrix}A & B\end{matrix}}\ \underline{\begin{matrix}E & F\end{matrix}}\ \begin{matrix}L & M & D\\ | & | & |\\ C & G & H\end{matrix}} \tag{4.8}$$

whence, implementing the identifications

$$\bar{L}^{\beta} \equiv A^{\beta}, \qquad \bar{M}^{\beta} \equiv B^{\beta}, \qquad \bar{G}_{\alpha} \equiv E_{\alpha}, \qquad \bar{H}_{\alpha} \equiv F_{\alpha} \tag{4.9}$$

we recover the result

$$I_1 = \frac{1}{3!\ \begin{matrix}F & E & D\\ | & | & |\\ A & B & C\end{matrix}} \tag{4.10}$$

We notice that (4.6) would also yield (4.10) if the pattern (3.8) along with the condition $C^{\mu}D_{\mu} = 0$ were used in conjunction with the kernel-letter identifications $L \equiv E$, $M \equiv F$, $G \equiv A$, and $H \equiv B$.

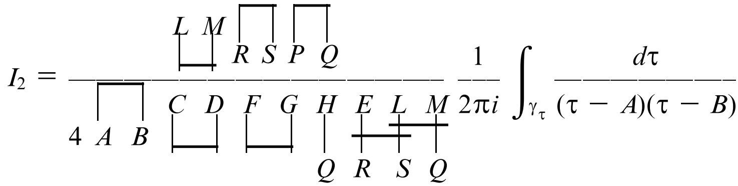
The diagram shown in Fig. 2 is expressed by the integral

$$I_2 = \frac{1}{(2\pi i)^9} \int_{\Gamma_{WXY}} \frac{\Delta WXY}{\begin{matrix}W & W & W & C & D & E & F & G & H & W\\ | & | & | & | & | & | & | & | & | & |\\ A & B & X & X & X & X & Y & Y & Y & Y\end{matrix}} \tag{4.11}$$

Adopting the parametrization prescription

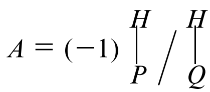
$$W_{\alpha} = L_{\alpha} + \mu M_{\alpha}, \qquad X^{\beta} = R^{\beta} + \nu S^{\beta}, \qquad Y^{\beta} = P^{\beta} + \tau Q^{\beta} \tag{4.12}$$

we can write



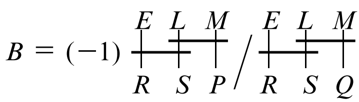
(4.13)

where



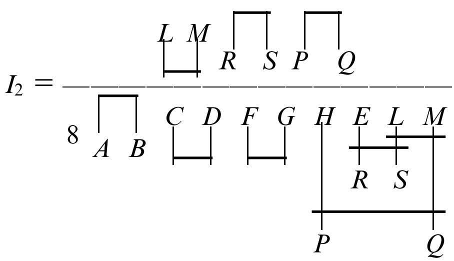
(4.14a)

and



(4.14b)

Consequently,

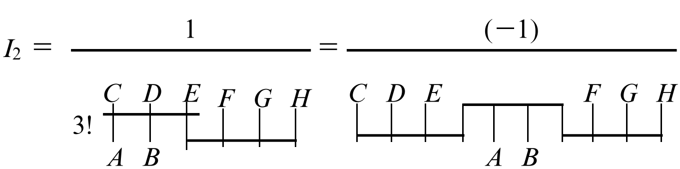


(4.15)

Hence, identifying

$$\overline{L}^{\beta} \equiv A^{\beta}, \qquad \overline{M}^{\beta} \equiv B^{\beta}, \qquad \overline{R}_{\alpha} \equiv C_{\alpha},$$
$$\overline{S}_{\alpha} \equiv D_{\alpha}, \qquad \overline{P}_{\alpha} \equiv F_{\alpha}, \qquad \overline{Q}_{\alpha} \equiv G_{\alpha} \tag{4.16}$$

and invoking (3.25), we obtain



(4.17)

Alternatively, (4.17) can be derived from (4.13) by combining (3.9) with the trivial identifications

$$L \equiv C, \qquad M \equiv D, \qquad R \equiv A, \qquad S \equiv B, \qquad P^{\beta} \equiv \overline{F}^{\beta}, \qquad Q^{\beta} \equiv \overline{G}^{\beta}$$

and taking E_{α} as a linear combination of F_{α}, G_{α} and H_{α}.

When calculating I_1 and I_2, one can make use of the Qadir kernels

$$K_W = \frac{1}{(2\pi i)^3}\int_{\Gamma_W}\frac{\Delta W}{\begin{matrix}W & W & W & W\\ | & | & | & |\\ T & V & K & J\end{matrix}} = \frac{1}{\begin{matrix}| & | & | & |\\ T & V & K & J\end{matrix}} \tag{4.18}$$

and

$$K_Y = \frac{1}{(2\pi i)^3}\int_{\Gamma_Y}\frac{\Delta Y}{\begin{matrix}I & N & U & V\\ | & | & | & |\\ Y & Y & Y & Y\end{matrix}} = \frac{1}{\begin{matrix}I & N & U & V\\ \llcorner & \perp & \perp & \lrcorner\end{matrix}} \tag{4.19}$$

which effectively constitute a holomorphic version of **RM**-separations. We point out that, according to our techniques, the evaluation leading to any such kernel involves the combination of ε-dualizations with the calculations of very simple integrals. For example,

$$K_Z = \frac{1}{2\pi i}\int_{\gamma_Z}\frac{\delta Z}{\begin{matrix}X\ Y & U & V\\ \llcorner\!\lrcorner & | & |\\ & Z & Z\end{matrix}} = \frac{\begin{matrix}\ulcorner\!\urcorner\\ X\ Y\end{matrix}}{2\begin{matrix}X\ Y & U\ V\\ \llcorner\!\lrcorner & X\ Y\end{matrix}} \tag{4.20}$$

For the diagram drawn in Fig. 3, we have the integral

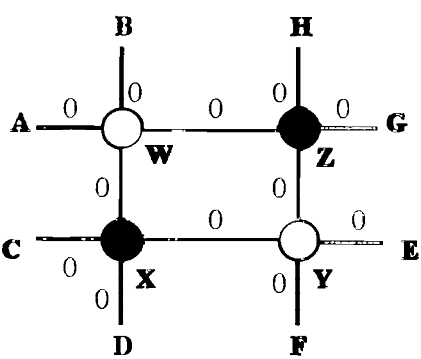


Fig. 3. A diagram with an internal box. All the singularity lines bear the number zero.

$$I_3 = \frac{1}{(2\pi i)^{12}} \int_{\Gamma_{WXYZ}} \frac{\Delta WXYZ}{\begin{matrix} W & W & W & W & C & D & Y & Y & Y & Y & G & H \\ | & | & | & | & | & | & | & | & | & | & | & | \\ A & B & X & Z & X & X & X & E & F & Z & Z & Z \end{matrix}} \tag{4.21}$$

Equations (4.18) and (4.19) can be utilized to carry out the *WYZ* integrals, say. This procedure leads us to

$$I_3 = \frac{1}{(2\pi i)^{3}} \int_{\Gamma_X} \frac{\Delta X}{\begin{matrix} & & & G & H & & & & C & D \\ \overline{|} & \overline{|} & \overline{|}\ \underline{\ } & | & | & \underline{\ }\ \overline{|} & \overline{|} & \overline{|} & | & | \\ A & B & X & & & X & F & E & X & X \end{matrix}} \tag{4.22}$$

Consequently, performing the integrals that involve the elementary inner products and parametrizing

$$X^\beta = P^\beta + \nu Q^\beta \tag{4.23}$$

we obtain

$$I_3 = \frac{\overline{P\quad Q}}{\underline{C\quad D}} \frac{1}{2\pi i} \int_{\gamma_\nu} \frac{d\nu}{A\nu^2 + B\nu + C} \tag{4.24}$$

where

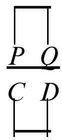
$$A = \begin{matrix} & & & G & H & & & \\ \overline{|} & \overline{|} & \overline{|}\ \underline{\ } & | & | & \underline{\ }\ \overline{|} & \overline{|} & \overline{|} \\ A & B & Q & & & Q & F & E \end{matrix} \tag{4.25a}$$

$$B = 2!\ \begin{matrix} & & & G & H & & & \\ \overline{|} & \overline{|} & \overline{|}\ \underline{\ } & | & | & \underline{\ }\ \overline{|} & \overline{|} & \overline{|} \\ A & B & | & & & | & F & E \\ & & \sim\sim & \sim\sim & \sim\sim & \sim\sim & & \\ & & P & & & Q & & \end{matrix} \tag{4.25b}$$

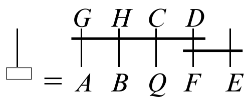
and

$$C = \begin{matrix} & & & G & H & & & \\ \overline{|} & \overline{|} & \overline{|}\ \underline{\ } & | & | & \underline{\ }\ \overline{|} & \overline{|} & \overline{|} \\ A & B & P & & & P & F & E \end{matrix} \tag{4.25c}$$

The contour γ_ν is supposedly taken to surround either of the singularities given by the zeros of the denominator of (4.24). We thus have

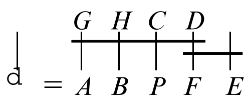
$I_3 =$  $(B^2 - 4AC)^{-1/2}$ (4.26)

At this point, it is useful to reexpress the coefficients (4.25) by making use of (3.34), (3.35), and (3.40). Hence, defining



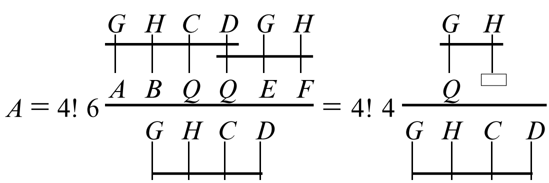
(4.27)

and



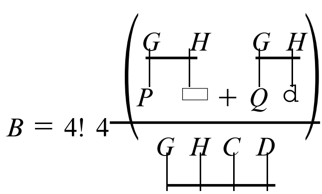
(4.28)

we write



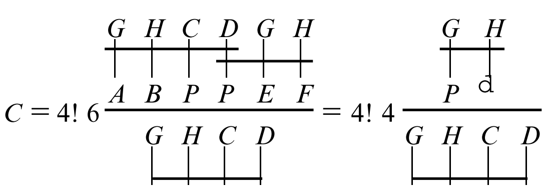
(4.29a)

along with



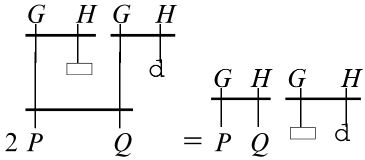
(4.29b)

and



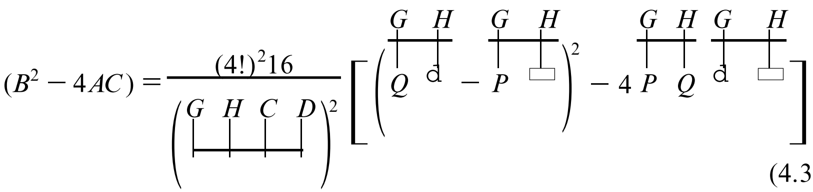
(4.29c)

After some manipulations implementing (3.6a) in the form



(4.30)

we thus obtain

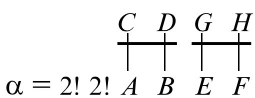


(4.31)

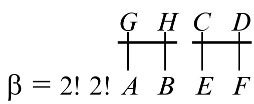
If the identifications $P^\beta \equiv \overline{C}^\beta \equiv R^\beta$ and $Q^\beta \equiv \overline{D}^\beta \equiv S^\beta$ are combined together with (4.23), the squared difference of (4.31) will be of the type of (3.45), whereas the skew kernel carrying (4.27) and (4.28) will turn out to be of the samed type as (3.41). Hence, putting into effect the formulas (3.42)–(3.44) together with (3.46) and (3.47) gives

$$I_3 = (\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma)^{-1/2} \tag{4.32}$$

where

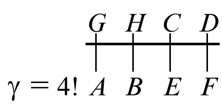


(4.33a)



(4.33b)

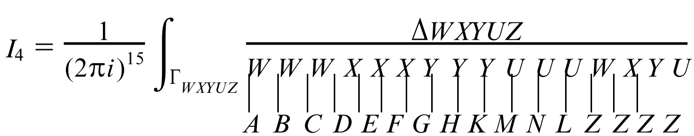
and



(4.33c)

We observe that (4.32) is invariant under the interchange $\alpha \leftrightarrow \beta$.

The integral



(4.34)

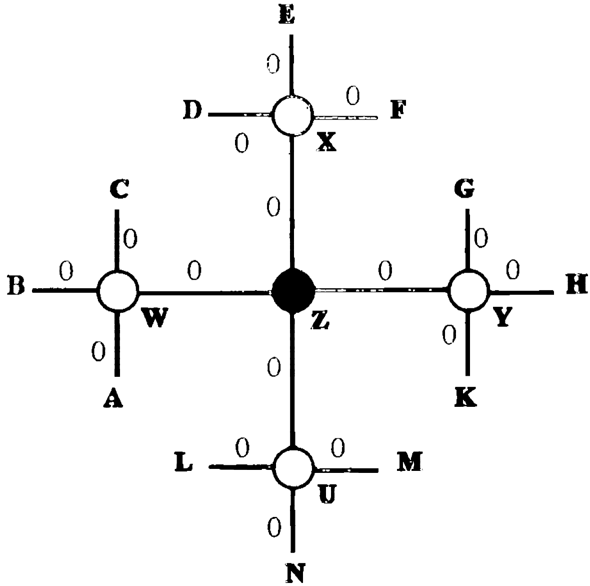


Fig. 4. A tree having five spotted vertices. The internal lines represent simple poles.

corresponds to the diagram depicted in Fig. 4. Rather than using (4.18) and (4.19) explicitly, we first perform integrations that lead to $\delta WXYU$ and then take up the prescription

$$W_\alpha = R_\alpha + \mu S_\alpha \tag{4.35a}$$

$$X_\alpha = T_\alpha + \nu V_\alpha \tag{4.35b}$$

$$Y_\alpha = P_\alpha + \tau Q_\alpha \tag{4.35c}$$

$$U_\alpha = I_\alpha + \sigma J_\alpha \tag{4.35d}$$

This alternative procedure yields the integral

$$I_4 = \frac{R\;S\;T\;V\;P\;Q\;I\;J}{2^4 \; A\;B\;D\;E\;G\;H\;M\;N} \frac{1}{(2\pi i)^3} \int_{\Gamma_Z} \frac{\Delta Z}{\begin{array}{cccccccc} R & S & T & V & P & Q & I & J \\ C & Z & F & Z & K & Z & L & Z \end{array}} \tag{4.36}$$

which leads us to the result

$$I_4 = \frac{\overbrace{R\,S}\,\overbrace{T\,V}\,\overbrace{P\,Q}\,\overbrace{I\,J}}{2^4\ \underbrace{A\ B}\ \underbrace{D\ E}\ \underbrace{G\ H}\ \underbrace{M\ N}}\ \frac{1}{\overset{R}{\underset{C}{|}}\ S\ \overset{T}{\underset{F}{|}}\ V\ \overset{P}{\underset{K}{|}}\ Q\ \overset{I}{\underset{L}{|}}\ J} \tag{4.37}$$

Therefore, adopting the dualization scheme

$$R_\alpha = \overline{A}_\alpha, \qquad S_\alpha = \overline{B}_\alpha, \qquad T_\alpha = \overline{D}_\alpha, \qquad V_\alpha = \overline{E}_\alpha$$
$$P_\alpha = \overline{G}_\alpha, \qquad Q_\alpha = \overline{H}_\alpha, \qquad I_\alpha = \overline{M}_\alpha, \qquad J_\alpha = \overline{N}_\alpha \tag{4.38}$$

we obtain

$$I_4 = \frac{1}{\overline{A\ B\ C}\ \ \overline{D\ E\ F}\ \ \overline{G\ H\ K}\ \ \overline{M\ N\ L}} \tag{4.39}$$

A diagram carrying a double pole is depicted in Fig. 5. The pertinent integral is

$$I_5 = \frac{1}{(2\pi i)^6} \int_{\Gamma_{WX}} \frac{\Delta WX}{\overset{W}{\underset{A}{|}}\ \overset{W}{\underset{B}{|}}\ \left(\overset{W}{\underset{X}{|}}\right)^2\ \overset{C}{\underset{X}{|}}\ \overset{D}{\underset{X}{|}}} \tag{4.40}$$

Lowering the pole order via W-integrations, say, and performing the simple-pole X-integrals, we get the useful expression

$$I_5 = \frac{1}{(2\pi i)^2} \int_{\gamma_{WX}} \frac{\delta WX}{\overline{A\ X}\ \underline{C\ D}\ \overset{W}{\underset{B}{|}}\ \overset{W}{\underset{X}{|}}} \tag{4.41}$$

which implies that

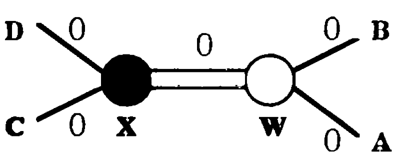


Fig. 5. A diagram carrying one double pole. The structure possesses an elementary-state character and a specific invariance property.

$$I_5 = \frac{1}{4}\,\frac{\begin{array}{cccc} L & M & & \\ \lfloor & \rfloor & K & J \end{array}}{\begin{array}{cccc} C\ D & L & M & \\ \lfloor\ \rfloor & | & | & \\ & & B & A \\ & K & & J \end{array}} \tag{4.42}$$

where L_α, M_α and K^β, J^β have been introduced to parametrize W_α and X^β, respectively. Implementing the identifications

$$L \equiv C, \qquad M \equiv D, \qquad K \equiv A, \qquad J \equiv B \tag{4.43}$$

and employing (3.5), we end up with the result

$$I_5 = \frac{1}{2!\begin{array}{cc} C & D \\ | & | \\ A & B \end{array}} \tag{4.44}$$

which is invariant under the simultaneous replacements

$$A \rightleftharpoons B, \qquad C \rightleftharpoons D \tag{4.45}$$

Owing to the invariance of the denominator of the integrand of (4.40) under the combination of reflections and kernel-letter relabelings, we would have obtained the same result if the WX-integrations giving rise to (4.41) had been the other way about. Clearly, (4.42) does not admit the implementation of dualizations.

The representative integral for the diagram drawn in Fig. 6 is written as

$$I_6 = \frac{1}{(2\pi i)^9}\int_{\Gamma_{WXY}} \frac{\Delta WXY}{\begin{smallmatrix} W \\ | \\ A \end{smallmatrix}\begin{smallmatrix} W \\ | \\ B \end{smallmatrix}\left(\begin{smallmatrix} W \\ | \\ X \end{smallmatrix}\right)^2\left(\begin{smallmatrix} Y \\ | \\ X \end{smallmatrix}\right)^2\begin{smallmatrix} Y \\ | \\ C \end{smallmatrix}\begin{smallmatrix} Y \\ | \\ D \end{smallmatrix}} \tag{4.46}$$

In contradistinction to (4.40), this integral provides a nonvanishing outcome regardless of the way in which we perform integrations yielding δ-forms. We have the connected expression

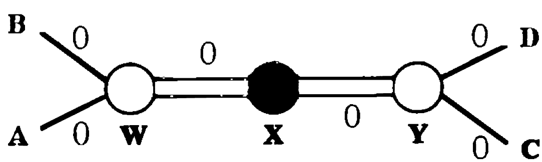


Fig. 6. One of the dual elementary-state diagrams with two double poles.

$$I_6 = \frac{1}{(2\pi i)^3} \int_{\gamma_{WXY}} \frac{\delta WXY}{W\ Y\ \ A\ B\ C\ D\ \ W\ Y\ X\ X} \tag{4.47}$$

Thus, parametrizing

$$W_\alpha = U_\alpha + \xi V_\alpha, \qquad X^\beta = R^\beta + \eta S^\beta \tag{4.48}$$

and carrying out the WX-integrals gives

$$I_6 = \frac{U\ V\ \ R\ S}{4\ \ A\ B\ C\ D} \frac{1}{2\pi i} \int_{\gamma_Y} \frac{\delta Y}{Y\ U\ V\ Y\ \ R\ S} \tag{4.49}$$

The denominator of the integrand of (4.49) is evidently proportional to the left-hand side of (3.9). Its structure can be made even more transparent if we use the dualization devices for R^β and S^β along with the conjugate of (3.39). We have, in effect,

$$I_6 = \frac{U\ V\ \bar{R}\ \bar{S}}{A\ B\ C\ D} \frac{1}{2\pi i} \int_{\gamma_Y} \frac{\delta Y}{U\ V\ Y\ \ Y\ \bar{R}\ \bar{S}} \tag{4.50}$$

To recover the result of (4.46) given earlier (Qadir, 1978), it suffices to implement the parametrization

$$Y_\alpha = L_\alpha + \tau M_\alpha \tag{4.51}$$

together with the prescription

$$L_\alpha \equiv \bar{A}_\alpha, \qquad M_\alpha \equiv \bar{B}_\alpha, \qquad U_\alpha \equiv \bar{C}_\alpha, \qquad V_\alpha \equiv \bar{D}_\alpha, \qquad R^\beta \equiv C^\beta \qquad S^\beta \equiv A^\beta \tag{4.52}$$

likewise taking account of (3.24c). We are then left with

$$I_6 = \frac{1}{A\ B\ C\ D} \tag{4.53}$$

In the sequel, we consider a couple of easy integrals carrying logarithms. The first is written as

$$I_7 = \frac{1}{(2\pi i)^3} \int_{\Gamma_W} \frac{\log\left(\begin{array}{c} W \\ | \\ P \end{array} \Big/ \begin{array}{c} W \\ | \\ Q \end{array} \right) \Delta W}{\begin{array}{c} W \\ | \\ A \end{array} \left(\begin{array}{c} W \\ | \\ B \end{array} \right)^2 \begin{array}{c} W \\ | \\ C \end{array}} \tag{4.54}$$

its diagrammatic form being given in Fig. 7. Performing the integrals that involve the simple poles, and parametrizing

$$W_\alpha = L_\alpha + \mu M_\alpha \tag{4.55}$$

yields

$$I_7 = \frac{\begin{array}{cc} L & M \\ \lfloor & \rfloor \end{array}}{\begin{array}{cc} \lceil & \rceil \\ A & C \end{array} \left(\begin{array}{c} M \\ | \\ B \end{array} \right)^2} \frac{1}{2\pi i} \frac{\partial}{\partial a} \int_{\gamma_\mu} \frac{\log\left(\begin{array}{c} W(\mu) \\ | \\ P \end{array} \Big/ \begin{array}{c} W(\mu) \\ | \\ Q \end{array} \right) d\mu}{(\mu - a)} \tag{4.56}$$

with a being taken to vary independently of μ, and the derivative being effectively evaluated at

$$a = (-1) \begin{array}{c} L \\ | \\ B \end{array} \Big/ \begin{array}{c} M \\ | \\ B \end{array} \tag{4.57}$$

Hence,

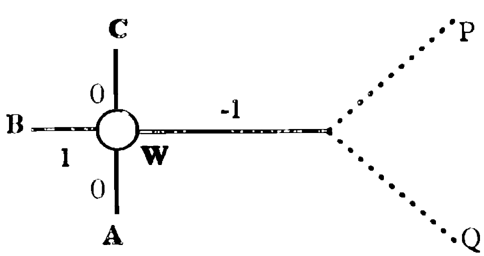


Fig. 7. A diagram carrying a negative number with a logarithmic branch.

$$I_7 = \frac{\overline{\underline{L\ M}}\ \ \begin{array}{cc} \overline{L\ \ M} \\ P\ \ Q \end{array}}{2\ \overline{\begin{array}{cc} A & C \end{array}}\ \begin{array}{cc} \overline{L\ \ M} \\ P\ \ B \end{array}\ \begin{array}{cc} \overline{L\ \ M} \\ Q\ \ B \end{array}} \tag{4.58}$$

Upon making the identifications

$$L_\alpha \equiv \overline{A}_\alpha, \qquad M_\alpha \equiv \overline{C}_\alpha \tag{4.59}$$

we obtain

$$I_7 = \frac{\overline{\begin{array}{cccc} P & Q & A & C \end{array}}}{\overline{\begin{array}{cccc} P & A & B & C \end{array}}\ \overline{\begin{array}{cccc} Q & A & B & C \end{array}}} \tag{4.60}$$

We next consider the integral

$$I_8 = \frac{1}{(2\pi i)^6} \int_{\Gamma_{WX}} \frac{\log\left(\begin{array}{c} W \\ | \\ X \end{array} \Big/ \begin{array}{cc} W & Q \\ | & | \\ P & X \end{array} \right) \Delta WX}{\left(\begin{array}{c} W \\ | \\ A \end{array} \right)^2 \begin{array}{cc} W & W \\ | & | \\ B & C \end{array} \left(\begin{array}{c} D \\ | \\ X \end{array} \right)^2 \begin{array}{cc} E & F \\ | & | \\ X & X \end{array}} \tag{4.61}$$

which corresponds to Fig. 8. Notice that the logarithmic functional dependence is invariant under rescalings of the variable twistors. Thus, carrying out the simple-pole integrations and parametrizing

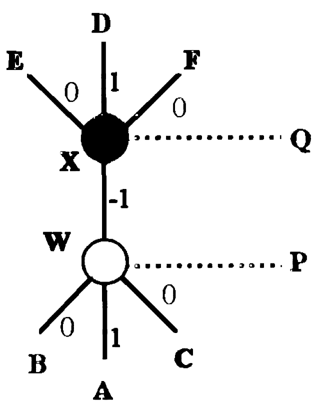


Fig. 8. A two-spotted-vertex diagram with a negative number and a logarithmic branch.

$$W_\alpha = L_\alpha + \mu M_\alpha, \qquad X^\beta = R^\beta + \nu S^\beta \tag{4.62}$$

gives

$$I_8 = \kappa \frac{\partial^2}{\partial a \partial b}\left[\frac{1}{(2\pi i)^2} \int_{\gamma_\mu \times \gamma_\nu} \frac{\log \left(\begin{matrix} W(\mu) \\ | \\ X(\nu) \end{matrix} \Big/ \begin{matrix} W(\mu) & Q \\ | & | \\ P & X(\nu) \end{matrix} \right) d\mu \wedge d\nu}{(\mu - a)(\nu - b)} \right] \tag{4.63}$$

where

$$\kappa = \frac{\begin{matrix} L \quad M \\ \sqcup \end{matrix} \begin{matrix} \sqcap \\ R \quad S \end{matrix}}{\begin{matrix} \sqcap \\ B \quad C \end{matrix} \begin{matrix} E \quad F \\ \sqcup \end{matrix} \begin{pmatrix} M \\ | \\ A \end{pmatrix}^2 \begin{pmatrix} D \\ | \\ S \end{pmatrix}^2} \tag{4.64}$$

In (4.63), the derivatives are evaluated at

$$a = (-1) \begin{matrix} L \\ | \\ A \end{matrix} \Big/ \begin{matrix} M \\ | \\ A \end{matrix} \tag{4.65a}$$

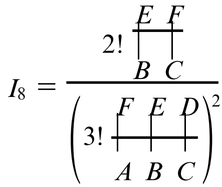
and

$$b = (-1) \begin{matrix} D \\ | \\ R \end{matrix} \Big/ \begin{matrix} D \\ | \\ S \end{matrix} \tag{4.65b}$$

We thus obtain the value

$$I_8 = \frac{\begin{matrix} L & M \\ \vdash & \dashv \\ R & S \end{matrix}}{8 \begin{pmatrix} L & M & D \\ \vdash & + & \dashv \\ A & R & S \end{pmatrix}^2} \tag{4.66}$$

whose denominator carries a connected kernel which is formally the same as the one borne by (4.8). Identifying L^β, M^β and R_α, S_α with B^β, C^β and E_α, F_α, respectively, yields the result

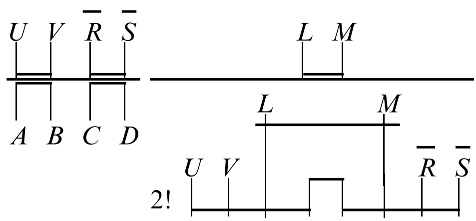


(4.67)

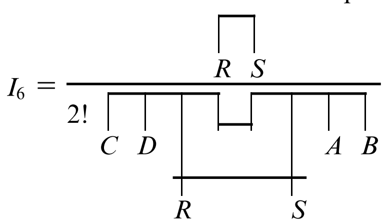
which is independent of P and Q. Of course, the combination of the identifications $L \equiv E$, $M \equiv F$, $R \equiv B$, and $S \equiv C$ with the condition $A^{\mu} D_{\mu} = 0$ also leads to (4.67).

5. CONCLUSIONS AND OUTLOOK

An interesting property of the integral (4.46) is concerned with the applicability of the Qadir-kernel procedure to the structure that arises when we just lower the pole orders. It is clear that the expression (4.50) amounts to



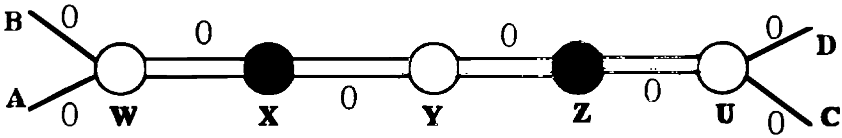
If we had in effect used (4.18) for integrating the *WY*-vertices of the lower order configuration, we would have obtained the equivalent result



with the entire class of admissible identifications being given by

$$
\begin{array}{ccc}
(LM) & (RS) & (UV) \\
(\overline{AB}, \overline{CD}) \leftarrow & (CA) & \rightarrow (\overline{CD}, \overline{AB}) \\
(\overline{AB}, \overline{CD}) \leftarrow & (DA) & \rightarrow (\overline{CD}, \overline{AB}) \\
(\overline{AB}, \overline{CD}) \leftarrow & (CB) & \rightarrow (\overline{CD}, \overline{AB}) \\
(\overline{AB}, \overline{CD}) \leftarrow & (DB) & \rightarrow (\overline{CD}, \overline{AB})
\end{array}
$$

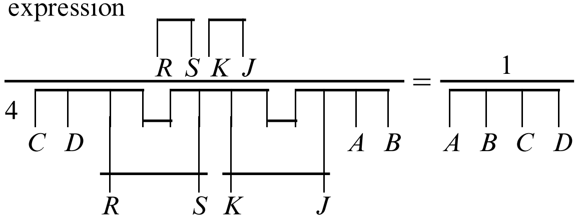
which also involves the interchange $(LM) \leftrightarrow (UV)$. Actually, this regularization property is borne by the (dual) double-pole elementary-state diagrams which carry an arbitrary odd number N of spotted vertices. The value of one of the generalized patterns is expressed as the coupling of $(N-1)/2$ blocks of the same type as those for I_6, the diagrams being thus related to one another through a white–black interchange rule. For example, for the diagram



with the parametrization prescription

$$X^{\beta} \to (R^{\beta}, S^{\beta}), \qquad Z^{\beta} \to (K^{\beta}, J^{\beta})$$

we have the expression



together with the kernel-letter scheme

$$(RS) \qquad (KJ)$$

$$(CA) \to (AD, BC, CD)$$

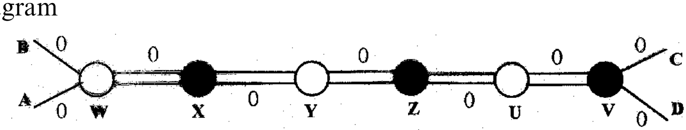
$$(DA) \to (AC, BD, CD)$$

$$(CB) \to (AC, BD, CD)$$

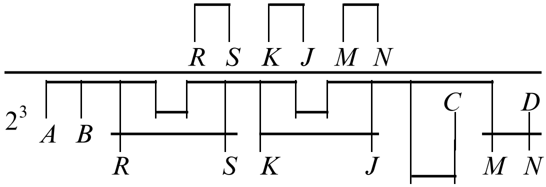
$$(DB) \to (AD, BC, CD)$$

the relations (3.38) and (3.39) certainly enabling us to interchange the roles of (RS) and (KJ).

On the other hand, the integral (4.40) does not admit the Qadir-kernel regularization as well insofar as the number of spotted vertices of the associated diagram is not large enough to ensure the achievement of a finite result. However, the invariance of I_5 under reflections and relabelings is carried over to the configurations that bear an even number $N \geq 4$. It turns out that the extended patterns are all regularizable, with either of the integrals for $N = 4$ admitting a consistent set of identifications. As an example, for the diagram



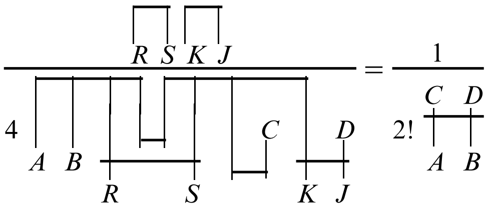
we have the value



where

$$X^\beta \rightarrow (R^\beta, S^\beta), \qquad Z^\beta \rightarrow (K^\beta, J^\beta), \qquad V^\beta \rightarrow (M^\beta, N^\beta)$$

For $N = 4$, we can thus write



with the identifications

$$(R^\beta, S^\beta) \equiv (\overline{C}^\beta, \overline{D}^\beta)$$
$$(K^\beta, J^\beta) \equiv (A^\beta, B^\beta)$$

Obviously, the number of elementary blocks now equals $(N - 2)/2$, the number of double lines being $(N - 1)$ in the case of either parity.

At first sight, one might think that sticking together simple and double poles would ultimately constitute a generally allowable procedure for lowering orders. In fact, such an integration procedure provides connected structures and still retains the regularizability when $N > 2$, but its implementation in the odd case makes one unable to recover the conformal invariance without taking up vanishing infinity-twistor inner products. For any even value of N, the order-lowering device that possesses $n = 1$ nevertheless shows up as a natural tool which has to be applied to only one of the double-pole branches of each of the pertinent diagrams. When $N = 2$ and $m = 3$, we may then apply that device twice to calculate either of the integrals for the internal-triple-line elementary states. We thus have

$$\frac{1}{(2\pi i)^6} \int_{\Gamma_{WZ}} \frac{\Delta WZ}{(A^\rho W_\rho)(Z^\mu W_\mu)^3 (Z^\sigma B_\sigma)} = \frac{1}{(A^\mu B_\mu)}$$

which contrasts with (4.40) particularly for admitting dualizations. It should be mentioned that the techniques developed here also provide tools for calculating twistor integrals other than the ones considered up to the present time. Among these nonordinary integrals is

$$\mathcal{J} = \frac{1}{(2\pi i)^6} \int_{\Gamma_{WZ}} \frac{F(W_\alpha Z^\beta)\Delta WZ}{(A^\rho W_\rho)(Z^\mu W_\mu)^m (Z^\sigma B_\sigma)}$$

from which one can derive in a somewhat elegant way the well-known statement

$$\frac{1}{(2\pi i)^6} \int_{\Gamma_{WZ}} \frac{\Delta WZ}{(W_\mu Z^\mu)^4} = 1$$

An important qualitative result we have obtained before is that the conformal symmetry borne by the classical twistor diagrams turns out to be reduced to the Poincaré symmetry when the evaluation procedures of Section 2 are carried through. Roughly speaking, what seems to be the crucial point as regards the above situation is that wedge products involving δ-forms are intimately tied in with the inner structure of the diagrams. It is this feature of the configurations which produces the symmetry reduction. Furthermore, it has been observed that the incorporation of suitable twistors into our parametrization schemes and the exclusion of spurious integration processes bring about a recovery of the conformal symmetry at least in the cases we have taken into consideration explicitly. Because of the nullity of the structures, meeting conditions have to be incorporated into the picture regardless of whether the identifications that yield standard patterns are prescribed in terms of conjugate twistors. In respect to this fact, it must be pointed out that integrations at Δ-form levels of internal simple-pole branches really provide acceptable results inasmuch as the pieces which destroy the holomorphicity of the corresponding integrands "evaporate" when dualizations like those of Section 3 are introduced into expressions of the form (4.20). We should emphasize, in addition, that the occurrence of divergent outcomes appears to be related to a breaking of Qadir kernels which essentially characterize connected simple-pole patterns. Thus the implementation of results coming from disconnecting internal simple lines seems to be forbidden. It may be hoped that such viewpoints will prove helpful to systematize the integration procedures once and for all.

It is worth remarking that our work can likewise be used to deduce a set of rules for directly building up twistor amplitudes from Feynman graphs without loops. On allowing for this latter situation, one may utilize the traditional procedure which includes starting with solutions of the relevant field equations and inserting them into the conventional expressions for the entries of the scattering matrix. The amplitudes appear as Poincaré-invariant wedge-product couplings adequately carrying the integrands of the universal contour integrals for interacting fields together with appropriate infinity-twistor factors and scaling invariant one-forms of the type $d\Phi/\Phi$, with Φ standing for the ordinary scalar product between variable twistors. The presence of such one-forms not only guarantees the strongly required independence between the twistors that partake of the main steps of the constructions, but also gives rise to a naive prescription for selecting spacetime vertices and makes feasible the appearance of basic diagrammatic pieces. Even when the scattering processes are taken to involve massive fields, it will be possible to sort out a procedure for fixing up kernel letters for auxiliary twistors and ascribe a physical meaning to meeting conditions. We expect that an investigation along these lines will supply a natural correspondence between spacetime and twistor configurations as well as a manifestly finite version of cross sections. Some of the philosophical aspects of twistor theory would be eventually strengthened.

ACKNOWLEDGMENTS

I would like to express my gratitude to Dr. Asghar Qadir for some discussions concerning the methods presented here. Part of this work was supported financially by the Brazilian agency CNPq.

REFERENCES

Hodges, A. (1983a). *Proceedings of the Royal Society London A,* **385**, 207.

Hodges, A. (1983b). *Proceedings of the Royal Society London A,* **386**, 185.

Hodges, A. (1983c). *Proceedings of the Royal Society London A,* **397**, 341.

Hodges, A., and Huggett, S. A. (1980). *Surveys in High Energy Physics,* **1**, 333.

Huggett, S. A. (1993). Cohomology and twistor diagrams, in *Twistors in Mathematics and Physics,* T. Bayley and R. Baston, eds., London Mathematical Society, p. 367.

Hughston, L. P. (1979). *Twistors and Particles,* Springer-Verlag, Berlin.

Penrose, R. (1967). *Journal of Mathematical Physics,* **8**, 345.

Penrose, R. (1975). Twistor theory: Its aims and achievements, in *Quantum Gravity, an Oxford Symposium,* C. J. Isham, R. Penrose, and D. W. Sciama, eds., Oxford University Press, Oxford.

Penrose, R., and MacCallum, M. A. H. (1972). *Physics Reports,* **6**, 241.

Penrose, R., and Rindler, W. (1986). *Spinor and Twistor Methods in Space-Time Geometry,* Cambridge University Press, Cambridge.

Qadir, A. (1971). Ph.D. thesis, Birkbeck College, London.

Qadir, A. (1978). *Physics Reports,* **39**, 167.

Sparling, G. (1975). Homology and twistor theory, in *Quantum Gravity, an Oxford Symposium,* C. J. Isham, R. Penrose and D. W. Sciama, eds., Oxford University Press, Oxford.